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Complex angular momenta and the Lorentz group

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Abstract. The unitary irreducible representations of the group SL(2C) are studied by taking the coupled states of two angular momenta J_1 and J_2 as basis functions. The unitarity of the representations leads to unphysical values of j_1 and j_2 such that one of the quantities $\sigma = j_1 + j_2 + 1$ and $j_{12} = j_1 - j_2$ is integral or half-integral and the other is purely imaginary or real. The formulae derived in this paper exhibit a beautiful symmetry with respect to the interchange of these two quantities. The basis functions are expressible in terms of terminating hypergeometric series, and, by using the properties of the latter, the matrices of the generators and of finite transformations are easily determined. The matrix element of the 'boost operator' corresponding to a pure Lorentz transformation in the $x_3 - x_4$ plane is found to take the form of a finite linear combination of ${}_3F_2$ functions.

1. Introduction

The investigation of the unitary representation of noncompact groups has engaged the attention of many authors (Gelfand and Naimark 1946, 1947, Gelfand *et al* 1963, Naimark 1964) for a long time and the discovery of the ever increasing number of elementary particles in recent years has served to direct attention to noncompact groups containing the Lorentz group as a subgroup. On the one hand such groups have been employed as the 'noninvariance' or 'dynamical' groups to generate mass spectra with infinitely many levels (Mukunda *et al* 1965, Kleinert 1967, Barut *et al* 1968) and recently to make predictions regarding other quantities of interest like the form factors (Barut and Kleinert 1967, Kuriyan and Sudarshan 1967). On the other hand wave equations which are invariant under the groups like the inhomogeneous de Sitter group and similar higher-dimensional groups have been systematically explored partly with the hope of incorporating internal symmetries alongside the space–time symmetry and partly as a way of presenting in unified form the system of Lorentz invariant wave equations (Bakri 1969, Fuschich and Krivsky 1968). Another problem of physical interest is the group-theoretical aspect of the Regge and Toller poles (Toller 1968) which are used in the phenomenological analysis of the scattering processes in elementary particle physics. It turns out that these may be identified with the poles of the Fourier transforms of certain distributions describing the scattering process. The invariance of the scattering amplitude under the inhomogeneous Lorentz group plays a role of central importance in all such investigations.

A good understanding of the properties of the group SL(2C) and its irreducible representations is therefore essential to all such investigations. The classical solution of this problem is that of Gelfand and Naimark (Naimark 1964) who were able to classify the unitary irreducible representations of SL(2C) in a basis in which the maximal compact subgroup SU(2) is reduced. They utilized a particular factorization

$$a = kz \quad \text{or} \quad a = ku, \qquad a \in \mathrm{SL(2C)}$$

$$k = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix} \in K, \qquad z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \in Z, \qquad u \in \mathrm{SU(2)}$$

and studied the representations in a Hilbert space $L^2(Z)$ of square integrable functions defined on the subgroup Z or SU(2). Although the Gelfand–Naimark approach is very powerful it is rather intuitive and the physical implications of the results are not immediately clear.

In this paper, however, we adopt a quite different standpoint and investigate the irreducible representations by using the usual machinery of spinor calculus which is familiar to any physicist. We start from the observation that in the spinor representation the operators of the subgroup SU(2) are identical with those of the coupling of two angular momenta J_1 and J_2. The basic state for the coupling of a pair of angular momenta which is expressible in terms of a hypergeometric function (Majumdar 1968) therefore formally serves as the basis functions for the representations of SL(2C) in which the SU(2) subgroup is reduced. The values of j_1 and j_2 which are fixed can be used to label the representation and are naturally related to the eigenvalues of the Casimir operators of the group. The action of the infinitesimal generators on the basic states can now be obtained easily using the recurrence relations satisfied by the hypergeometric functions.

We next proceed to examine the problem of the unitary representations and show that the unitarity of the representation leads to complex values of the angular momenta j_1 and j_2 such that, either: (*a*) $j_{12} = j_1 - j_2$ is integral or half-integral and $\sigma = j_1 + j_2 + 1$ is pure imaginary or real; or (*b*) σ is an integral or half-integral negative number and j_{12} is pure imaginary or real. Of course the weights of the SU(2) representations contained in SL(2C), that is, the j values do remain physical and we recover the principal or the complementary series of the representation according as one of the numbers σ and j_{12} is pure imaginary or real while the other is an integer or a half-integer. In our approach therefore the problem of the unitary representations of SL(2C) leads us to a formal analytic continuation of the standard coupling problem to unphysical complex values of the angular momenta subject to certain restrictions depending upon the nature of the representation.

The unitary representation $a \to Va$ is now realized in the space of these coupled SU(2) basis functions and the problem of finite transformation is solved easily by using standard expansions in terms of hypergeometric functions. The matrix element for finite transformation can be obtained as a finite linear combination of ${}_3F_2$ functions. From this we recover the elementary spherical function of the representation of the principal and complementary series as appropriate special cases.

2. Infinitesimal operators and the basis functions of the representation

The group SL(2C) is the group of all 2×2 complex matrices with unit determinant:

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \det a = 1. \tag{2.1}$$

The six infinitesimal operators denoted by J and F generate the Lie algebra of SL(2C)

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k; \qquad [J_i, F_j] = \mathrm{i}\epsilon_{ijk}F_k$$
$$[F_i, F_j] = -\mathrm{i}\epsilon_{ijk}J_k \tag{2.2}$$

where J are the generators of the SU(2) subgroup and F those for pure Lorentz transformations. These generators can be represented as differential operators in the space of the analytic functions $f(\xi_1, \xi_2; \eta_1, \eta_2)$ where (ξ_1, ξ_2) and (η_1, η_2) are spinors transforming according to the fundamental representation (2.1) and its complex conjugate respectively.

For future applications it is convenient to consider,

$$\begin{aligned} J_+ &= J_1 + \mathrm{i}J_2 = \left(\eta_1\frac{\partial}{\partial\eta_2} - \xi_2\frac{\partial}{\partial\xi_1}\right) \\ J_- &= J_1 - \mathrm{i}J_2 = \left(\eta_2\frac{\partial}{\partial\eta_1} - \xi_1\frac{\partial}{\partial\xi_2}\right) \\ J_3 &= \left\{-\frac{1}{2}\left(\xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2}\right) + \frac{1}{2}\left(\eta_1\frac{\partial}{\partial\eta_1} - \eta_2\frac{\partial}{\partial\eta_2}\right)\right\} \end{aligned} \tag{2.3}$$

$$\begin{aligned} F_+ &= F_1 + \mathrm{i}F_2 = \mathrm{i}\left(\xi_2\frac{\partial}{\partial\xi_1} + \eta_1\frac{\partial}{\partial\eta_2}\right) \\ F_- &= F_1 - \mathrm{i}F_2 = \mathrm{i}\left(\xi_1\frac{\partial}{\partial\xi_2} + \eta_2\frac{\partial}{\partial\eta_1}\right) \\ F_3 &= \frac{\mathrm{i}}{2}\left\{\left(\xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2}\right) + \left(\eta_1\frac{\partial}{\partial\eta_1} - \eta_2\frac{\partial}{\partial\eta_2}\right)\right\}. \end{aligned} \tag{2.4}$$

An inspection of the generators given by (2.3) now reveals that if we make a simple change of variables

$$\xi_1 = \zeta_2; \qquad \xi_2 = -\zeta_1 \tag{2.5}$$

then the operators of the SU(2) subgroup become formally identical with those for the coupling of a pair of angular momenta J_1 and J_2. The coupled states ϕ_m^j may therefore serve as the basis functions for the representation of SL(2C) which is explicitly reduced with respect to the SU(2) subgroup.

The nature of the coupling mentioned in the preceding paragraph can be understood by considering the linear combinations

$$X_i = \tfrac{1}{2}(J_i + \mathrm{i}F_i), \qquad Y_i = \tfrac{1}{2}(J_i - \mathrm{i}F_i).$$

The two sets of operators X and Y commute with one another and the operators in each set satisfy the commutation relations of angular momentum. This is evident from equation (2.2) and also from the differential expressions (2.3) and (2.4). In their differential forms X_i involve ξ_1, ξ_2 only and Y_i involve η_1, η_2 only. X and Y, therefore, represent some kind of angular momenta which may be coupled to form the resultant J. But, as they are not hermitian, they do not generate SU(2) groups or any noncompact versions of SU(2) and their eigenvalues are not required to be non-negative integers or half-integers. In the following we shall allow arbitrary values of j_1, j_2 to be determined later from the condition of unitarity of the representations.

The coupled SU(2) basis function which has been obtained by one of us (Majumdar 1968) in an earlier paper is expressible in terms of a hypergeometric function (Bailey 1935, Erdélyi 1953) and can be written as

$$\phi_m^j = (-1)^m N_{jm} \xi_1^{j_1-j_2-m} \xi_2^{j_1+j_2+m} \eta_2^{2j_2} \left(1+\frac{\xi_1\eta_1}{\xi_2\eta_2}\right)^{j_1+j_2-j}$$
$$\times {}_2F_1\left(-j-m, j_1-j_2-j; -2j; 1+\frac{\xi_1\eta_1}{\xi_2\eta_2}\right) \tag{2.6}$$

where

$$N_{jm} = \{(j+m)!(j-m)!\}^{-1/2}.$$

It is now necessary to determine the action of the operators F on this function. We first note that the operators $\mp F_\pm, \sqrt{2}F_3$ are the $\pm 1, 0$ components of a vector operator A_1 in the SU(2) subspace. We are therefore led to consider the reduced matrix elements

$$R_{j',j} = \langle j' \| A_1 \| j \rangle$$

corresponding to $j' = j-1, j$ and $j+1$. It also follows that the matrix elements of $F_\pm$ and F_3 will be nonvanishing only if $m' = m \pm 1$ and m, respectively.

The evaluation of the reduced matrix elements is greatly facilitated by use of the recurrence relations satisfied by the hypergeometric functions:

$$F(a,b,c) = \frac{a(a+1)b(b-c)}{c^2(c^2-1)} x^2 F(a+2, b+1, c+2)$$
$$+F(a, b-1, c-2) - \frac{a(2b-c)}{c(c-2)} xF(a+1, b, c)$$

$$x\frac{\partial F(a,b,c)}{\partial x} = \frac{-a(a+1)b(b-c)(b-c+1)}{c(c^2-1)(2b-c)} x^2 F(a+2, b+1, c+2)$$
$$-\frac{b(c-2)}{2b-c} F(a,b,c) + \frac{b(c-2)}{2b-c} F(a, b-1, c-2) \tag{2.7}$$

$$(1-x)F(a,b,c) = \frac{b(b-c)(c-a)(c-a+1)}{c^2(c^2-1)} x^2 F(a, b+1, c+2) + \frac{(c-a)(2b-c)}{c(c-2)}$$
$$\times xF(a-1, b, c) + F(a-2, b-1, c-2)$$

$$(1-x)\frac{\partial F(a,b,c)}{\partial x} = (a-c+b)F(a,b,c) + \frac{(a-c)(b-c)}{c} F(a-1, b, c)$$
$$+\frac{b(b-c)(a-c)(c-a+1)}{c^2(c+1)} xF(a, b+1, c+2)$$

$$(1-x)F(a,b,c) = \frac{b(b-c)(c-a)(c-a-1)}{c^2(c^2-1)} x^2 F(a+1, b+1, c+2) + \frac{(c-a-1)(2b-c)}{c(c-2)}$$
$$\times xF(a,b,c) + F(a-1, b-1, c-2)$$

$$(1-x)\frac{\partial F(a,b,c)}{\partial x} = \frac{ab}{c} F(a,b,c) + \frac{ab(b-c)(c-a)}{c^2(c+1)} xF(a+1, b+1, c+2).$$

These recurrence relations follow after a straightforward calculation from those given by Erdélyi *et al* (Erdélyi 1953).

The transformations induced on the basis functions by the operators (2.4) follow immediately from the above equations when one sets $a = -j-m, b = j_1-j_2-j$, $c = -2j$ and lead to

$$R_{j-1,j} = \tfrac{1}{2}\mathrm{i}\{2j(2j+1)\}^{1/2}\alpha_j(\sigma+j)$$

$$R_{j,j} = -\frac{2\mathrm{i}\sigma j_{12}}{\{2j(j+1)\}^{1/2}} \tag{2.8}$$

$$R_{j+1,j} = -2\mathrm{i}\{(2j+1)(2j+2)\}^{1/2}(\sigma-j-1)$$

where

$$\sigma = j_1+j_2+1; \qquad j_{12} = j_1-j_2; \qquad \alpha_j = \frac{j^2-j_{12}^2}{j^2(4j^2-1)}. \tag{2.9}$$

From these equations it is obvious that by repeated application of F_3 or $F_\pm$ to ϕ^j_m it is possible to get all the states corresponding to $j_0, j_0+1, \ldots j, j+1 \ldots$, where j_0 is the lowest weight of the SU(2) representation contained in SL(2C). The ladder must therefore be truncated at the bottom and the coefficients of $\phi^{j_0-1}_m$, that is, R_{j_0-1,j_0}, should vanish. Thus,

$$(\sigma+j_0)(j_0^2-j_{12}^2) = 0 \tag{2.10a}$$

whence

$$j_{12} = \pm j_0 \tag{2.10b}$$

or

$$\sigma = -j_0. \tag{2.10c}$$

These solutions actually lead to sets of equivalent representations. A representation in the space of the basis functions corresponding to the first solution, as we shall see presently, is identical with that of Naimark and covers all possible unitary irreducible representations of the group. The second solution on the other hand provides us with an altogether new scheme leading to an identical set of representations.

To find irreducible representations we must find the invariant subspaces of the operators $J_\pm, J_3$ and $F_\pm, F_3$. First it is clear that every invariant subspace is characterized by a unique value of σ and j_{12}. In other words σ and j_{12} are invariants and must be fixed in an irreducible representation. It is therefore natural to find that these quantities are related to the eigenvalues of the Casimir operators:

$$\begin{aligned} C_1\phi^j_m &= \{F_+F_-+F_-F_++2F_3^2-(J_+J_-+J_-J_++2J_3^2)\}\phi^j_m \\ &= -2(\sigma^2+j_{12}^2-1)\phi^j_m \\ C_2\phi^j_m &= (J_+F_-+J_-F_++F_+J_-+F_-J_++4J_3F_3)\phi^j_m \\ &= -4\mathrm{i}\sigma j_{12}\phi^j_m. \end{aligned}$$

Two representations $D(j_{12}, \sigma)$ and $D(j'_{12}, \sigma')$ are equivalent provided the Casimir operators are the same for both and the spectra of the diagonal generators are the same. Thus the representations $D(j_{12}, \sigma)$ and $D(-j_{12}, -\sigma)$ are equivalent. The Casimir operators also remain invariant under the interchange of σ and j_{12}. This is, in fact, revealed by the second solution which just interchanges the role of σ and j_{12}. The basis functions in these two cases are however different. In the first case when $j_{12} = \pm j_0$, the parameters of the hypergeometric function in (2.6) are negative integers while for $\sigma = -j_0$ the parameter 'b' becomes complex, and these two bases do not seem to be connected by any simple transformation.

3. The unitary representation

The representation found above can be made unitary simply by introducing appropriate normalizers in (2.6). Let

$$f^j_m = a_j \phi^j_m \tag{3.1}$$

be a basis for the unitary representation. We introduce a suitable inner product in the Hilbert space of these vectors such that

$$(f^j_m, f^{j'}_{m'}) = \delta_{jj'}\delta_{mm'}. \tag{3.2}$$

The unitarity of the representation now requires

$$(F_3 f^j_m, f^{j'}_{m'}) = (f^j_m, F_3 f^{j'}_{m'}). \tag{3.3}$$

Setting $m' = m$ and $j' = j$ and $j-1$ respectively we get,

$$j_{12}\sigma = -j^*_{12}\sigma^* \tag{3.4a}$$

$$\left|\frac{a_j}{a_{j-1}}\right|^2 = \frac{4(j^2-\sigma^{*2})j^2(4j^2-1)}{|j+\sigma|^2(j^2-j^2_{12})}. \tag{3.4b}$$

We first investigate the consequences of the solution (2.10b). When $j_{12} = \pm j_0$ the equation (3.4a) reads,

$$j_0\sigma = -j_0\sigma^*.$$

This is possible only in the following two cases: (a) j_0 arbitrary so that σ is pure imaginary; (b) $j_0 = 0$ so that σ is arbitrary. In this case all the three parameters of the hypergeometric function are negative integers.

When σ is pure imaginary (say, $\sigma = \mathrm{i}\rho/2$), the set of all representations corresponding to all possible pairs $(j_0, \frac{1}{2}\mathrm{i}\rho)$ is the principal series of the representations. When $j_0 = 0$ and σ is real we have

$$\left|\frac{a_j}{a_{j-1}}\right|^2 = \frac{4(j^2-\sigma^2)(4j^2-1)}{(j+\sigma)^2}. \tag{3.5}$$

This expression must be a positive definite quantity for all $j = 1, 2 \ldots$. Obviously this is possible only if $0 < \sigma^2 < 1$. However since $D(0, \sigma)$ and $D(0, -\sigma)$ are equivalent it is sufficient to consider $D(0, \sigma)$ with $0 < \sigma < 1$ which is the so called complementary series of the representations.

When $\sigma = -j_0$ in accordance with (2.10c) the equation (3.4a) again leads to two distinct possibilities: (a) j_0 arbitrary so that $j_{12} = -\frac{1}{2}\mathrm{i}\rho$; ($b$) $j_0 = 0$ so that j_{12} is arbitrary. In this case, however, the parameter $b = j_{12} - j$ of the hypergeometric function is a complex number.

The first possibility (*a*) again gives rise to the principal series and the second possibility (*b*) to the complementary series of the representations. For the complementary series $\sigma = -j_0 = 0$ and

$$\left|\frac{a_j}{a_{j-1}}\right|^2 = \frac{j^2(4j^2-1)}{j^2-j_{12}^2}.$$

The positive definiteness of this expression again ensures that $0 < j_{12}^2 < 1$.

Either of the above set of representations is of course equivalent to the previous set and is obtainable from the same by an interchange of σ and j_{12}.

A solution to the recurrence relation (3.4*b*) can now be easily obtained and is given by,

$$a_j = e^{i\pi j}\left(\frac{\Gamma(2j+2)\Gamma(2j+1)\Gamma(j-\sigma+1)}{\Gamma(j-j_{12}+1)\Gamma(j+j_{12}+1)\Gamma(j+\sigma+1)}\right)^{1/2}. \tag{3.6}$$

This normalization, apart from a trivial phase factor, is equivalent to the one obtained by one of us (Majumdar 1968) in a different connection for physical values of j_1 and j_2.

Using (3.6) we now obtain the unitary irreducible representation of the group SL(2C) corresponding to a given pair (σ, j_{12}) in the space of the basis functions f^j_m. These are given by the formulae,

$$\begin{aligned} J_\pm f^j_m &= \{(j\mp m)(j\pm m+1)\}^{1/2} f^j_{m\pm 1} \\ J_3 f^j_m &= m f^j_m \end{aligned} \tag{3.7a}$$

while the actions of the operators $F_\pm, F_3$ are determined by the normalized reduced matrix elements

$$\begin{aligned} T_{jj} &= R_{jj} \\ T_{j-1,j} &= \frac{a_j}{a_{j-1}} R_{j-1,j} = -\{2j(2j+1)\}^{1/2} C_j \\ T_{j,j-1} &= \frac{a_{j-1}}{a_j} R_{j,j-1} = -\{2j(2j-1)\}^{1/2} C_j \end{aligned} \tag{3.7b}$$

where

$$C_j = \frac{i}{j}\left(\frac{(j^2-\sigma^2)(j^2-j_{12}^2)}{4j^2-1}\right)^{1/2}.$$

It is interesting to observe that the coefficients C_j are invariant under the interchange of σ and j_{12}. Therefore the matrix elements of finite transformations should also be invariant under this interchange.

Since the coefficients $C_j \neq 0$ for $j > j_0$ either for the principal or the complementary series of the representations, the above formulae show that the representation contains all the weights $j = j_0, j_0+1, j_0+2\ldots$ and is therefore infinite dimensional.

If we further note that the basis functions, as defined by (2.6), (3.1) and (3.6), are homogeneous functions of degree $2j_1 = \sigma + j_{12} - 1$ in (ξ_1, ξ_2) and $2j_2 = \sigma - j_{12} - 1$ in (η_1, η_2), we can easily obtain the operators for the unitary representation. Writing

$$z = \frac{\xi_1}{\xi_2}, \qquad \bar{z} = \frac{\eta_1}{\eta_2} \tag{3.8}$$

and using Euler's theorem we easily obtain

$$
\begin{aligned}
F_{+} &= \mathrm{i}\frac{\partial}{\partial z}-\mathrm{i}\bar{z}^{2}\frac{\partial}{\partial \bar{z}}+\mathrm{i}\bar{z}(\sigma-j_{12}+1)\\
F_{-} &= -\mathrm{i}z^{2}\frac{\partial}{\partial z}+\mathrm{i}\frac{\partial}{\partial \bar{z}}+\mathrm{i}z(\sigma+j_{12}-1)\\
F_{3} &= \mathrm{i}z\frac{\partial}{\partial z}+\mathrm{i}\bar{z}\frac{\partial}{\partial \bar{z}}-\mathrm{i}(\sigma-1)\\
J_{+} &= -\frac{\partial}{\partial z}-\bar{z}^{2}\frac{\partial}{\partial \bar{z}}+\bar{z}(\sigma-j_{12}-1)\\
J_{-} &= z^{2}\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}-z(\sigma+j_{12}-1)\\
J_{3} &= -z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial \bar{z}}+j_{12}.
\end{aligned}
\tag{3.9}
$$

It should be pointed out that (3.9) represents two sets of operators defined in the spaces of two distinct basis functions corresponding to the solutions $j_{12} = \pm j_0$ and $\sigma = -j_0$.

We now proceed to realize the unitary irreducible representations of the group in the Hilbert space of these basis functions:

$$f^{j}_{m} = \xi_2^{j_{12}+\sigma-1}\eta_2^{-j_{12}+\sigma-1}\Psi^{j}_{m}(z,\bar{z}) \tag{3.10}$$

where

$$\Psi^{j}_{m}(z,\bar{z}) = a_j N_{jm} z^{j_{12}-m}(1+z\bar{z})^{\sigma-j-1}{}_2F_1(-j-m, j_{12}-j; -2j; 1+z\bar{z}). \tag{3.11}$$

For every matrix $a \in \mathrm{SL}(2C)$ we define a corresponding operator V_a in the space of these functions such that,

$$V_a f(\xi_i;\eta_i) = f(\xi'_i;\eta'_i) \tag{3.12}$$

with

$$\xi'_i = \sum_{j=1}^{2} a_{ji}\xi_j; \qquad \eta'_i = \sum_{j=1}^{2} a^{*}_{ji}\eta_j.$$

Using (3.10) and (3.12) we can easily obtain the form taken by the operator V_a for this realization of the space. A straightforward calculation leads, without any difficulty, to

$$V_a\Psi^{j}_{m}(z,\bar{z}) = (a_{12}z+a_{22})^{j_{12}+\sigma-1}(a^{*}_{12}\bar{z}+a^{*}_{22})^{-j_{12}+\sigma-1}\Psi^{j}_{m}\left(\frac{a_{11}z+a_{21}}{a_{12}z+a_{22}},\frac{a^{*}_{11}\bar{z}+a^{*}_{21}}{a^{*}_{12}\bar{z}+a^{*}_{22}}\right). \tag{3.13}$$

If we accept the solution (2.10*b*) namely

$$j_{12} = \tfrac{1}{2}r, \qquad j_0 = |\tfrac{1}{2}r|$$

where r is a positive or a negative integer the equation (3.13) is identical with the one given by Naimark. The solution $\sigma = -j_0$ on the other hand provides us with an alternative realization of the representation in a different basis.

4. Finite transformation matrix elements

One advantage in our approach based on the multispinors is that it allows us in a straightforward manner to evaluate the matrix elements of finite transformations.

It is well known that every matrix $a \in \mathrm{SL(2C)}$ can be represented in the form

$$a = u_1 \epsilon u_2 \tag{4.1}$$

where u_1, u_2 are unitary unimodular matrices corresponding to space rotation and

$$\epsilon = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \tag{4.2}$$

which corresponds to pure Lorentz transformation in the x_3–x_4 plane. It is therefore sufficient to consider the matrix elements

$$d_{jj'}^{mm',(j_{12},\sigma)}(\epsilon) = (\Psi_{m'}^{j'}, V_\epsilon^{(j_{12},\sigma)} \Psi_m^j) \tag{4.3}$$

of the boost operator.

From (3.13) it follows that

$$V_\epsilon^{(j_{12},\sigma)} \Psi_m^j(z, \bar{z}) = a_j N_{jm} (1-\zeta)^{\frac{1}{2}(j_{12}-m-\sigma+1)} z^{j_{12}-m} S_m^j \tag{4.4}$$

where

$$\begin{aligned} S_m^j &= (x+\zeta-x\zeta)^{\sigma-j-1} {}_2F_1(-j-m, j_{12}-j;\, -2j;\, x+\zeta-x\zeta) \\ x &= 1+z\bar{z}, \qquad \zeta = 1-\epsilon^4. \end{aligned} \tag{4.5}$$

For $j_{12} = \frac{1}{2}r, |\frac{1}{2}r| = j_0$, the hypergeometric function appearing above will terminate at $(j-j_{12})$ or $(j+m)$ according as $j_{12}+m \gtrless 0$ and this is simply related to the Jacobi polynomials to be defined separately in these two domains. The connection with Jacobi polynomials can be easily seen by reversing the hypergeometric series. Writing $a = -j-m, b = j_{12}-j, c = -2j$, we have, for $b-a > 0$,

$${}_2F_1(a, b; c; x) = (-1)^b \frac{(-a)!(b-c)!}{(-c)!(b-a)!} x^b {}_2F_1\left(b, 1-c+b; 1-a+b; \frac{1}{x}\right) \tag{4.6}$$

and for $j_{12} = j_0$ this is a multiple of the Jacobi polynomial

$$P_n^{(\alpha,\beta)}(1-2y) = \binom{n+\alpha}{n} {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; y) \tag{4.7}$$

with

$$\begin{aligned} n &= -b = j-j_0, \qquad \alpha = b-a = j_0+m > 0 \\ \beta &= a+b-c = j_0-m, \qquad y = \frac{1}{x}. \end{aligned}$$

The situation corresponding to $j_{12} = -j_0$ can be obtained easily using the permutation symmetry of the hypergeometric functions and, as can be readily checked, leads to identical results.

If we expand the hypergeometric series in (4.5) and note that binomial expansions can be defined even for complex powers we have (for $j_0+m > 0$),

$$S^j_m = \sum_{n=0}^{j-j_0}\sum_{\lambda=0}^{\infty}(-1)^n\frac{(j+m)!(j-j_0)!(2j-n)!\Gamma(j-\sigma-n+\lambda+1)(1-\zeta)^{\sigma-j+n-1}}{(j+m-n)!(j-j_0-n)!(2j)!n!\lambda!\Gamma(j-\sigma-n+1)} \times\left(\frac{\zeta}{\zeta-1}\right)^{\lambda}x^{\sigma-j-\lambda+n-1}. \tag{4.8}$$

From the well known formula (Erdélyi *et al* 1953),

$$y^{\rho} = \sum_{l=0}^{\rho}\frac{(\alpha+\rho)!(2l+\alpha+\beta+1)(l+\beta+\alpha)!(-\rho)_l}{(l+\alpha+\beta+\rho+1)!(\alpha+l)!}P_l^{(\alpha,\beta)}(1-2y) \tag{4.9}$$

and using (4.6) and (4.7) it is now easy to obtain the expansion,

$$x^{-(j-j_0+\lambda-n)} = \sum_k\frac{(j+m+\lambda-n)!(2k+1)!(j-j_0+\lambda-n)!x^{j_0-k}}{(j+\lambda-n+k+1)!(j+\lambda-n-k)!(k-j_0)!(k+m)!} \times {}_2F_1(-k-m, j_0-k;\, -2k;\, x). \tag{4.10}$$

The above equation in conjunction with (4.4), (4.5) and (4.8) now leads to,

$$V_\epsilon\Psi^j_m(z) = a_jN_{jm}(1-\zeta)^{\frac{1}{2}(j_0-m+\sigma-2j-1)}\sum_n\sum_\lambda\sum_k(-1)^n\frac{(j+m)!(j-j_0)!(2j-n)!}{(j+m-n)!(j-j_0-n)!n!\lambda!(2j)!} \times\frac{\Gamma(j-\sigma-n+\lambda+1)(j+m+\lambda-n)!(2k+1)!(j-j_0+\lambda-n)!(1-\zeta)^n}{\Gamma(j-\sigma-n+1)(j+\lambda-n+k+1)!(j+\lambda-n-k)!(k-j_0)!(k+m)!} \times\left(\frac{\zeta}{\zeta-1}\right)^{\lambda}\frac{1}{a_kN_{km}}\Psi^k_m(z). \tag{4.11}$$

This formula is obtained under the restriction $j_0+m > 0$. However, as can be readily verified, the same result holds for $j_0+m < 0$, the upper limit of the sum over 'n' being $(j+m)$.

From this we can easily calculate the boost matrix elements (4.3) which after some calculations reduce to a linear combination of ${}_3F_2$ functions. It is easily seen that the formula holds for $j_{12} = \pm j_0$ and is given by,

$$d^{mm',(j_{12},\sigma)}_{jj'}(\epsilon) = (-1)^{j-j'}\frac{\delta_{mm'}a_jN_{jm}\Gamma(j-\sigma+1)}{a_{j'}N_{j'm'}(2j')!}\epsilon^{2(j_{12}-m+\sigma-2j'-1)} \times\sum_{n=0}^{j-j_{12}}\frac{(j+m)!(j-j_{12})!}{(j+m-n)!(j-j_{12}-n)!}\frac{(2j-n)!}{(j'-j+n+1)!\Gamma(j-\sigma-n+1)} \times{}_3F_2\left(\begin{matrix}j'-\sigma+1, j'+m+1, j'-j_{12}+1\\ j'-j+n+1, 2j'+2\end{matrix};1-\epsilon^{-4}\right)\quad \text{for } j'\geqslant j. \tag{4.12}$$

For $j' < j$ an identical formula holds but the range of summation will be different; in that case $j-j'\leqslant n\leqslant j-j_{12}$. It is interesting to note that the equation (4.12) is symmetric in σ and j_{12}. This ensures that the solutions (2.10*b*) and (2.10*c*) lead to identical representations.

From this we can easily recover the elementary spherical function of the representation of the principal and complementary series as appropriate special cases. These are, in fact, matrix elements corresponding to the most degenerate case,

$$j' = j = m = m' = j_0 = 0.$$

Under this condition the summation drops out and the ${}_3F_2$ function degenerates into the ordinary ${}_2F_1$ function. For the representation of the principal series we have,

$$d_{00}^{00,(0\rho)}(\epsilon) = \epsilon^{2(\frac{1}{2}i\rho-1)}{}_2F_1(1-\tfrac{1}{2}i\rho, 1; 2; 1-\epsilon^{-4}).$$

On using the standard integral representation,

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t$$

we immediately get

$$d_{00}^{00,(0\rho)}(\epsilon) = \frac{2}{\rho}\frac{\sin\rho\tau}{\sinh 2\tau}$$

where $\epsilon = \mathrm{e}^{-\tau}$. This is the so called elementary spherical function of the representation of the principal series and agrees exactly with that obtained by Naimark in an entirely different manner. It is evident from our calculation that the corresponding quantity for the complementary series of the representation can be obtained simply by replacing ρ by $i\rho$.

5. Conclusion

The present analysis shows that the unitary irreducible representations of SL(2C) can be realized in terms of an analytic continuation of the coupled SU(2) basis functions to unphysical complex domains. Our result is similar to that of Barut and Fronsdal (1966) who showed that the unitary representations of SU(1, 1) can be realized in a space of basis functions which are analytic continuations of the usual SU(2) basis. It is interesting to note that the hypergeometric series in the expression (2.6) for the basis function retains its terminating character even in the case of SL(2C) group. This enables us to directly carry the techniques of the rotation group to the case of SL(2C).

Acknowledgments

We wish to thank the referee for pointing out that our technique (§ 2) can be used to study the O(3, 1) symmetry of the hydrogen atom.

Appendix

The techniques of § 2 of this paper can be used to study the O(3, 1) symmetry of the hydrogen atom. It is well known that the bound states of the nonrelativistic hydrogen atom form the bases of irreducible representations of the group O(4) of rotations in four dimensions. The larger symmetry is intimately connected with the so called 'accidental degeneracy' of the energy levels in a Coulomb field with respect to the values of l and is attributable to the existence of an additional operator

$$M = (2\mu)^{-1}(p \times L - L \times p) - Ze^2 r/r \tag{A.1}$$

which commutes with the hamiltonian $H = (2\mu)^{-1}p^2 - Ze^2/r$. The irreducible representations of O(4) that occur in the hydrogen problem are, however, of the special type $D^{j_1 j_2}$ for which $j_1 = j_2 = \frac{1}{2}(n-1)$ (Bander and Itzykson 1966). For a fixed energy, the usual angular momentum states $|nlm\rangle$ and the Stark states $|n_1 n_2 m\rangle$ obtained by separating the wave equation in parabolic coordinates are different realizations of the same irreducible representation of O(4). The two sets of states are connected by a linear transformation whose coefficients are the SU(2) Clebsch–Gordan coefficients (Park 1960, Hughes 1967):

$$\{\tfrac{1}{2}(n-1), \tfrac{1}{2}(n-1), l; \tfrac{1}{2}(m+n_2-n_1), \tfrac{1}{2}(m+n_1-n_2), m\}. \tag{A.2}$$

In the case of continuum states belonging to the positive spectrum the hermitian operators L and $\hat{M} = (\mu/2E)^{1/2}M$ satisfy the commutation relations

$$[L_i, L_j] = \mathrm{i}\epsilon_{ijk}L_k, \qquad [L_i, \hat{M}_j] = \mathrm{i}\epsilon_{ijk}\hat{M}_k, \qquad [\hat{M}_i, \hat{M}_j] = -\mathrm{i}\epsilon_{ijk}L_k \tag{A.3}$$

and, hence, build up the Lie algebra of the homogeneous Lorentz group O(3, 1). The principal quantum number n now takes the purely imaginary value $-\mathrm{i}N$ (where $N = -\mu Ze^2/k$, $k = \mathrm{i}p_0 = (2\mu E)^{1/2}$), while j_1 and j_2 again become equal to $\frac{1}{2}(n-1)$. It is, therefore, expected that the matrix elements between the Stark and the angular momentum states of the continuum will be Clebsch–Gordan coefficients of the type (A.2) generalized for complex values of j_1 and j_2. To test this point we consider the particular solution

$$\psi_c = \mathrm{e}^{ikz}{}_1F_1(-\mathrm{i}N, 1, ik(r-z)) \tag{A.4}$$

which represents the scattering of an electron by a point charge $+Ze$ at the origin. An application of Hughes's (1967) operators for $\frac{1}{2}(L_i \pm \mathrm{i}\hat{M}_i)$ to this function gives

$$j_1 = j_2 = -m_1 = m_2 = \tfrac{1}{2}(n-1). \tag{A.5}$$

Thus, the Stark and the angular momentum states can be represented as

$$|j_1 j_2, m_1 m_2\rangle = |\tfrac{1}{2}(n-1)\tfrac{1}{2}(n-1), -\tfrac{1}{2}(n-1)\tfrac{1}{2}(n-1)\rangle$$

and $|nl0\rangle$, respectively. The complex Clebsch–Gordan coefficients connecting these states can be obtained from the general expression (2.6) and are found to have the value (Majumdar 1968, equations (4) and (6))

$$\begin{aligned}
&\{\tfrac{1}{2}(n-1), \tfrac{1}{2}(n-1), l; -\tfrac{1}{2}(n-1), \tfrac{1}{2}(n-1), 0\} \\
&\quad = (-1)^{n-l-1}(l!)^{-1}C_{\frac{1}{2}(n-1),\frac{1}{2}(n-1),l}\Gamma(n)_2F_1(-l, -l; -2l; 1) \\
&\quad = (-1)^{n-l-1}\Gamma(n)(2l+1)^{1/2}\{\Gamma(n-l)\Gamma(1+l+n)\}^{-1/2}.
\end{aligned} \tag{A.6}$$

It should, therefore, be possible to expand ψ_c in a series of the form

$$\begin{aligned}
\psi_c = \sum_l &(-1)^{n-l-1}\Gamma(n)(2l+1)^{1/2}\{\Gamma(n-l)\Gamma(1+l+n)\}^{-1/2} \\
&\times N_{nl}\rho^l \mathrm{e}^{-\rho/2}{}_1F_1(1+l-n, 2l+2, \rho)P_l(\cos\theta)
\end{aligned} \tag{A.7}$$

where $\rho = -2ikr$. That is indeed the case, as is seen by introducing appropriate normalizers N_{nl} for the angular momentum states such that the representations of O(3, 1) become unitary. But for an l independent factor which remains undetermined the expansion then takes the form given in the standard text books (Messiah 1961).

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